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Evolution model with a cumulative feedback coupling

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The paper is concerned with a toy model that generalizes the standard Lotka-Volterra equation for a certain population by introducing a competition between instantaneous and accumulative, history-dependent nonlinear feedback the origin of which could be a contribution from any kind of mismanagement in the past. The results depend on the sign of that additional cumulative loss or gain term of strength λ. In case of a positive coupling the system offers a maximum gain achieved after a finite time but the population will die out in the long time limit. In this case the instantaneous loss term of strength u is irrelevant and the model exhibits an exact solution. In the opposite case $\lambda<0$ the time evolution of the system is terminated in a crash after t_s provided $u=0$. This singularity after a finite time can be avoided if $u\neq 0$. The approach may well be of relevance for the qualitative understanding of more realistic descriptions.

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I. INTRODUCTION

A certain amount of money $p(t)$ available for a financial transaction depends apparently on the history of the sample to which it belongs. A certain species of a population at time t should also depend on the population at previous times. Furthermore, the time evolution of money or any species is governed by nonlinearities that tend to avoid an unrestricted growing up. The interplay between birth and death processes results in a finite stationary solution. Because the time evolution of $p(t)$ is determined by local time gain and loss processes the stationary value is reached after an infinite time interval. But a more real situation seems to wait for a finite time to get a maximum gain (or loss). After that time the population proceeds further to evolute in time. To model such a situation the evolution equation has to change in a significant manner. Generally the time evolution of $p(t)$ is characterized by simultaneous competitive terms, i.e., at each time t the change of the quantity $p(t)$ is balanced by gain and loss terms at the same time t. As the result of the competition the system develops a finite stationary solution. In the present paper we extend the well-known Lotka-Volterra model by including an accumulative coupling of the momentary evolution to all the values taken in the past. Thus the time evolution of $p(t)$ at time t is coupled to all former events within the interval $0<t'<t$. This can be realized by constrained rates that depend on the quantity $p(t)$ in a nonlinear cumulative manner. Such a huge yield of a certain transaction or debts in the past should lead to a modified evolution at present time. Guided by the aim to enhance the present asset the amount of relevant money changes in time. To take the history dependence literally the evolution equation for $p(t)$ has to include a memory term that reflects the way by which an initial asset has been accumulated, for instance, by rates of interest, the yield, the business on the stock market. It means in other words that the changing rate of money at time t is also determined by the accumulation of capital at a former time $t'<t$ or simpler by the evolution within the whole interval $[0,t]$. The owner of the capital is, in general, interested to augment $p(t)$. Regardless the fluctuations the capital is subjected to the present availability of money and depends in a decisive manner on both the instantaneous amount of money and on the gain or the loss of capital in the past. Hence, it seems to be worth studying a financial model under inclusion of a feedback coupling. In the present paper we propose a model in which such a retardation effect is taken into account that can be, moreover, studied within an analytical approach. The model yields a nontrivial result assuming that the memory kernel depends on the amount of money itself. Obviously, such an assumption is rather realistic because the investment of capital in the past is strongly influenced by the accessible money at that time. Consequently, the problem will be formulated in a self-consistent manner. Furthermore, the time evolution may be controlled by a permanent coupling to the initial value of $p(t=0)$ denoted by p_0.

Our model can be grouped into the increasing interest in applying concepts and methods of statistical physics to study problems of biological evolution, for a recent review see [1], the financial market [2–4] and other complex systems from heartbeats to weather [5], to politics [6], to medical care [7], and further to ecology [8]. Similar to statistical physics the mentioned systems, such as economic ones, consist of a large number of interacting units (agents). Hence, experiences gained by studying complex physical systems might yield new results in economics. However, agents making financial transactions are thinking units, the interactions of which are not quantified in detail. Consequently, economic systems are quite different and much more complex. Nevertheless, the evolution of financial data should be governed by laws and methods well known in statistical mechanics. Apparently, various financial time series undergo random processes as particles making Brownian motions. Hence, one of the reasons to analyze financial systems by methods developed for physical systems is the challenge of understanding the dynamics of a strongly fluctuating system with a large number of interacting elements [9,10]. Moreover, simple models are discussed whose origin lies in market scenario [11] and they yield an unusual type of microdynamics of more general interest. Another approach related directly to our intentions consists of modeling the dynamics of money directly [12], the dynamics of which is studied by an evolution equation completely different to our one.

In the present paper we introduce an analytical model for a certain amount of money $p(t)$ available for one owner. This money is a part of the total capital flow and, therefore, subjected to a lot of influences that cannot be specified in detail. Comparing with the situation in statistics the last quantity corresponds to the many particle distribution function. Our interest is, however, devoted to a single "particle" function. Following the well established projector formalism due to Ref. [13], see also [14], the reduced evolution equation should offer an additonal memory kernel as successfully demonstrated in deriving a nonlinear evolution equation of Fokker-Planck type [15] the form and the relevance of which are discussed by analytical [15] and numerical methods [16]. Notice that the approach had been very fruitful in studying the freezing processes in glasses [17,18]. We believe that memory effects are a feature of dynamical complex systems.

II. THE CUMULATIVE FEEDBACK COUPLING

The model considered is an extension of the Lotka-Volterra model under the inclusion of a cumulative feedback term. The simplest version is defined by

$$\partial_t p(t) = r p(t) - u p^2(t) - \lambda p(t) \int_0^t p(t_1) dt_1 . \qquad (1)$$

The first term corresponds to a gain term, the second one is a loss term provided $u>0$. As demonstrated in the following section the model can be solved exactly in case of $u=0$. The last term describes a kind of meticulous memory. All changes in the past such as mutations of the species or a climate change leading to desertification or overfertilization resulting in a salinization have an influence of the growth or death rate. In the same sense a wrong financial investment or a miscarried speculation contribute in a cumulative manner to the present disposable amount of money. The model had been discussed before in Ref. [19] to illustrate various mathematical methods that may be applied to solve such a nonlinear model. Here, we want to discuss the physical realization and different special cases not so far considered before. In particular, we study both cases $\lambda>0$ as well as $\lambda<0$.

If $\lambda=0$ the model can be even solved exactly. In the long time limit the model yields a finite stationary value

$$\lim_{t\to\infty} p(t) \equiv p_s = \frac{r}{u} \quad \text{if} \quad \lambda = 0. \qquad (2)$$

In the general case the model can be analyzed introducing the variable

$$R(t) = |\lambda| \int_0^t dt_1 p(t_1). \qquad (3)$$

The basic Eq. (1) can be written in a phase space representation as

$$\partial_t p(t) = p(t)[r - u p(t) \mp R(t)],$$
$$\partial_t R(t) = |\lambda| p(t), \qquad (4)$$

where the upper sign corresponds to $\lambda>0$. Apparently in this case the memory term is the counterpart to the growth rate r. For the further discussion it seems to be more appropriate to use the set of Eqs. (4). Using those equations one can easily find a differential equation for $p(R)$, the solution of which reads

$$p(R) = \left(p_0 - p_s \mp \frac{|\lambda|}{u^2}\right) \exp\left(-\frac{u}{|\lambda|} R\right) + \left(p_s \pm \frac{|\lambda|}{u^2} \mp \frac{R}{u}\right), \qquad (5)$$

where the upper sign corresponds to a positive cumulative parameter $\lambda>0$. The set of coupled Eqs. (4) or the separatrix (5) can be used to derive some general predictions for the complete model that will be discussed in the subsequent section. In the limting case $u\to 0$ the separatrix is simply a parabola

$$p(R) = p_0 + \frac{r}{|\lambda|} R \mp \frac{1}{2|\lambda|} R^2 \qquad (6)$$

with $p_0 = p(t=0)$. The case $u=0$ exhibts an exact solution presented in the following section.

III. RESULTS

A. The case $u=0$

Firstly let us consider the case when the instantaneous loss term is absent, i.e., $u=0$. The solution of Eq. (1) with the initial value p_0 is

$$p(t) = p_0 \frac{(1+A)^2 e^{t/\tau}}{(1+A e^{t/\tau})^2} \quad \text{with} \quad \tau = \frac{1}{\sqrt{r^2 + 2\lambda p_0}},$$
$$A = \frac{1 - r\tau}{1 + r\tau}. \qquad (7)$$

Note that the result can be also extended to a more general memory term of the kind $R(t) = |\lambda| \int_0^t dt_1 p^{\mu}(t_1)$ with an exponent $\mu \geqslant 1$. In that case the solution reads

$$p(t) = p_0 \frac{(1+A)^{2/\mu} e^{t/\mu\tau}}{(1+A e^{t/\tau})^{2/\mu}}$$

with modified parameters A and τ. The feature of this simplified version of Eq. (1) with $u=0$ is a direct coupling of the instantaneous population $p(t)$ to its initial value p_0. This coupling is further manifested in the two different cases discussed now.

(i) If $r\tau<1$, the parameter A is positive that is realized in case of $\lambda p_0>0$. In Fig. 1 the analytical solution is depicted as the solid line whereas the dotted line denotes the numerical solution of the evolution equation. The slight discrepancy is due to the still rather large discretization interval. Starting

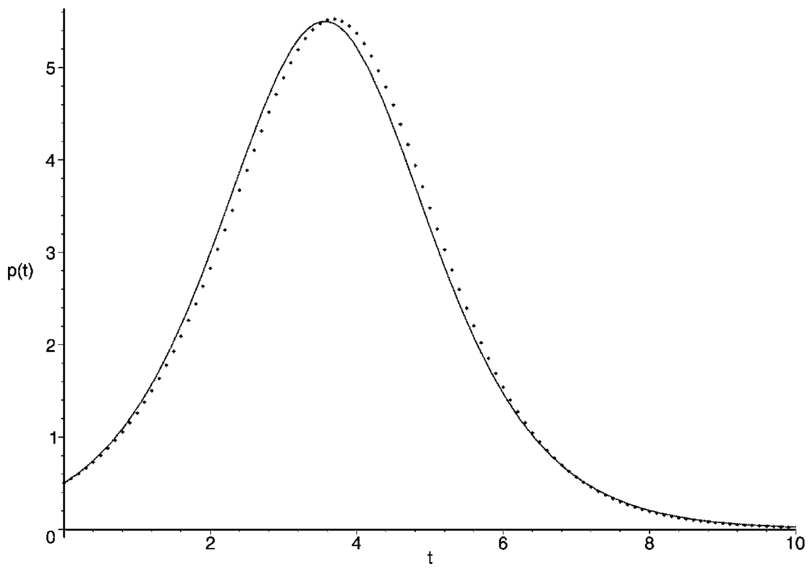

FIG. 1. Analytical solution (solid line) of Eq. (1) with $u=0$ and $p(0)=0.1$ vs a numerical solution (dotted line) with time-steps width $\Delta t=0.1$. All quantities are dimensionless.

with a finite and positive initial value $p_0>0$ the system reaches a maximum after a finite time

$$t_m=\frac{1}{\sqrt{r^2+2\lambda p_0}}\ln\left[\frac{\sqrt{r^2+2\lambda p_0}+r}{\sqrt{r^2+2\lambda p_0}-r}\right]. \tag{8}$$

The maximum gain is

$$p_m\equiv p(t_m)-p_0=\frac{r^2}{2\lambda}. \tag{9}$$

After reaching the maxima the cumulative term offers its toxic effect. The number of the population is decreased to the initial value after $2t_m$. In the long time limit $t\to\infty$ the population becomes extinct (or the whole money is lost). The bad conditions summed up from the initial time $t=0$ up to the instantaneous time t favors the extinction. Notice that there is no slowing down of the relaxation time.

(ii) If $r\tau>1$, realized for $\lambda p_0<0$ the system tends to a singular behavior after the finite time $t_s\equiv t_m$. The evolution results in a crash after that finite time interval provided $p_0>0$. Whereas unfavorable external conditions with $\lambda>0$ lead to the extinction the opposite case gives rise to the singularity at which the evolution has to stop. Remark that the results presented here for $u=0$ can be derived alternatively using the separatrix $p(R)$ given by Eq. (6) . To that aim $p(R)$ is inserted into the second Eq. (4). As the solution we get $R(t)$ that is anew inserted in Eq. (4). From here it results $p(t)$ in accordance with Eq. (7). In particular, the singularity observed for negative λ is obtained from Eq. (4) by

$$t_s=\frac{1}{|\lambda|}\int_0^\infty dt_1[p(t_1)]^{-1}\equiv t_m. \tag{10}$$

In case of a positive λ the corresponding equation has no real solution, i.e., there is no singular behavior in accordance with Eq. (8).

B. Additional loss term $u\neq 0$

In this section we discuss the solution of the general model defined by Eq. (1). For $u\neq 0$ the coupled set of Eqs. (4) or alternatively the separatrix $p(R)$ Eq. (5), can be used to analyze the solution. The phase space portrait is presented in Fig. 2. The arrows indicate that the solutions tend to the separatrix. As suggested by the special case $u=0$ we have to distinguish a postive or negative coupling strength λ. In particular, for $\lambda<0$ the cumulative term is a competitive one in comparison to the nonlinear u term in Eq. (1). In case $\lambda>0$ there occurs a maximum $p(R_e)$ at

$$R_e=\frac{\lambda}{u}\ln\left[1+\frac{u^2}{\lambda}(p_s-p_0)\right].$$

In case of a negative cumulative coupling $\lambda<0$ it results a minimum. Both cases can be summarized to

$$p(R_e)=p_s\left(1\mp\frac{|\lambda|}{ru}\ln\left[1\pm\frac{u^2}{|\lambda|}(p_s-p_0)\right]\right). \tag{11}$$

If $\lambda>0$ and $p_s>p_0$, the extremum is always positive. The maximum gain is, however, smaller than the stationary solution p_s, i.e., $p_0<p(R_e)<p_s$. In Fig. 3 we show the influence of the loss term up^2 in Eq. (1). As discussed above the population $p(t)$ reaches a maximum and tends to zero in the infinite time limit. The generic behavior is identical to that for $u=0$. Insofar, the parameter u is irrelevant. However, some details as the height of the maximum and the time after that the maximum is reached depends on u. The larger u the smaller is the maximum gain. The numerical results depicted in Fig. 3 are in accordance with the analytical approach because we find $p(R_e)<p_m$ where p_m is given by Eq. (9). In particular, based on Eq. (5), it results for small u

$$p(R_e)=p_m-\frac{ur}{\lambda}\left[p_0+\frac{r^2}{3\lambda}\right]+O(u^2).$$

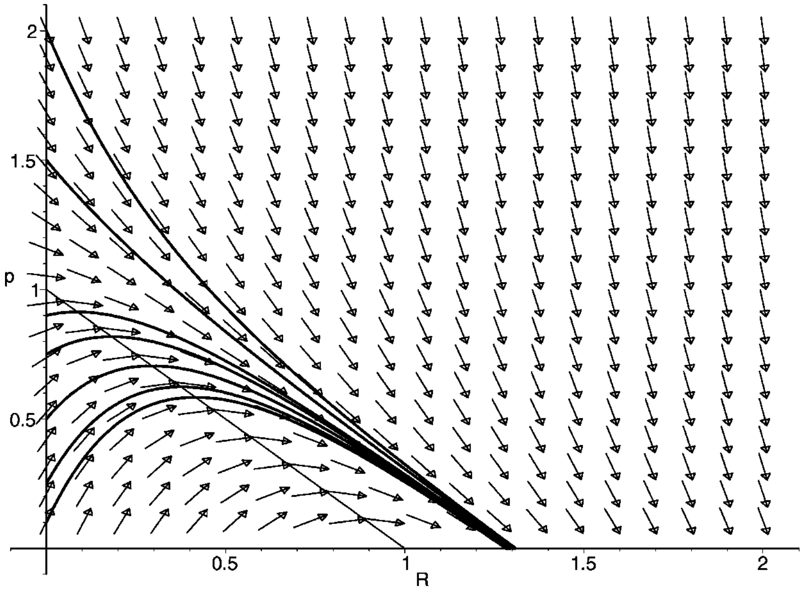

FIG. 2. Phase space portrait in the p-R plane with $ru/\lambda=3$.

Furthermore, the maximum is reached after a shorter time interval $t_m(u)<t_m(u=0)$. In Fig. 4 the role played by the cumulative loss term $\lambda>0$ is analyzed. Whereas for $\lambda=0$ the stationary value, obtained for $t\to\infty$ is simply $p_s=r/u$. Already an arbitrary small cumulative coupling $\lambda>0$ leads to a toxic stationary solution $p(t\to\infty)=0$. Any kind of mismanagment in the past gives rise to a decay of $p(t)$. The maximum value of the amount of money decreases with an increasing positive cumulative coupling. This result is also in accordance with the analytical findings. As discussed in Sec. III A a negative feedback $\lambda<0$ exhibits a crash after a finite time interval. This situation reminds to the resonance catastrophy for small vibrations. As in that case the catastrophy can be prevented by including a friction term the role of which is adopted by the term $up^2(t)$ in Eq. (1). Because Eq. (1) is equivalent to a second order differential equation, compare also Eq. (14) it is appropriate to introduce the first integral (energy) by the relation

$$E=\frac{1}{2}\left(\frac{\dot{p}}{p}\right)^2\pm|\lambda|p=\frac{1}{2}\dot{w}^2+\lambda e^w \quad \text{with} \quad w(t)=\ln p(t). \tag{12}$$

We find

$$\frac{dE}{dt}=-F \quad \text{with} \quad F=u\frac{\dot{p}^2}{p} \tag{13}$$

using the complete basis equation (1). The additional loss term plays the role of a dissipative term. The corresponding equation of motion can be solved and is given in terms of the Lambert W function. Let us further discuss the behavior in case of $\lambda<0$ and $u\neq 0$. In Fig. 5 we show that the crash time t_s, see Eq. (10), is shifted to infinity. The slope of the curves are proportional to $|\lambda|/u$, i.e., the population develops an essential singularity with respect to the parameter u. It results

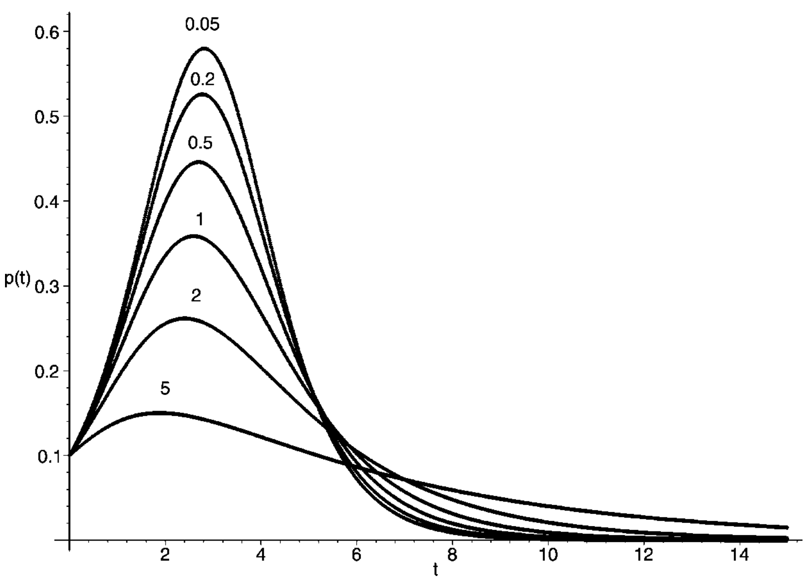

FIG. 3. Numerical solution of Eq. (1) with fixed dimensionless parameter $\lambda=1$ and different values $u=0.05;0.2;0.5;1.0;2.0;5.0$.

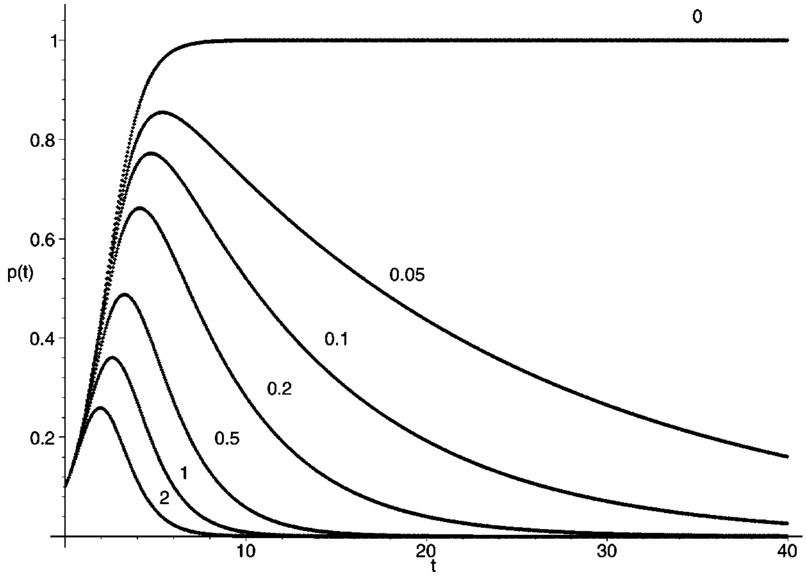

FIG. 4. Numerical solution of Eq. (1) with fixed parameter $u=1$ and different values $\lambda=0;0.05;0.1;0.2;0.5;1.0;2.0$ in dimensionless units.

$p(t)\propto \exp(|\lambda|t/u)$. The numerical result can be confirmed by an analytical approach. Starting with Eq. (1) or alternatively with Eq. (13) the basis equation reads in terms of the variable $w(t)=\ln p(t)$

$$\frac{d^2w}{dt^2}+e^w\left[u\frac{dw}{dt}+\lambda\right]=0. \tag{14}$$

An asymptotic solution is obviously $\dot{w}=-\lambda/u$. A linear stability analysis reveals that the solution is stable if $\lambda<0$. From here we conclude the asymptotic behavior

$$p(t)\propto \exp\left(\frac{|\lambda|}{u}t\right)$$

according to Eq. (5). In Fig. 5 the function $w(t)$ is presented for different values of the parameter u. Whereas in case $u=0$ a singular behavior is observed in line with the anayltical result, an arbitrary small value of u shifts the singularity to $t_s\rightarrow\infty$. Therefore, in case of a negative cumulative feedback coupling the u term is strongly relevant. This behavior is apparently because for $\lambda<0$ there is only a single loss term proportional to p^2.

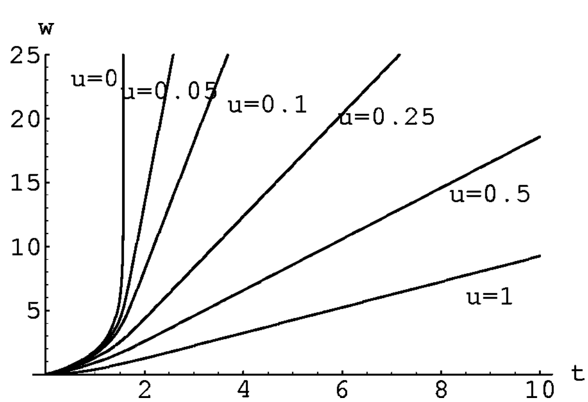

FIG. 5. $w(t)=\ln p(t)$ for negative $\lambda=-1$ and different paramters u. The slope is given by λ/u.

Notice that one can verify the essential results of this paper using a discrete version of our basis Eq. (1). To that aim let us replace $p(t)$ by p_n where $t=n\Delta t$. Eq. (1) is rewritten as

$$p_{n+1}=(r+1)p_n-(u+\lambda)p_n^2-\lambda p_n(p_0+p_1+\cdots p_{n-1}). \tag{15}$$

The fixed point of this equation reads for large n

$$p^*\simeq\frac{r}{u+\lambda n\Delta t}.$$

For $\lambda>0$ we get $p^*\equiv 0$ for $n\rightarrow\infty$. In case $\lambda=0$ it results the conventional stationary value r/u. If $\lambda<0$, the system reveals a singularity after $n_s=u/|\lambda|\Delta t$.

IV. CONCLUSIONS

In this paper we have made an attempt to study the time evolution of a certain population available as initial population p_0 under the influence of an additional cumulative memory term. However, there are also other motivations from ecology, economy, and politics for such a kind of model. Thus, the flow of the money is not only determined by the actual input and output of the asset but in an essential way by the accumulation rate at a previous time. Likewise in an election the vote is not only determined by some instantaneous decisions but by a kind of feeling accumulated by decisions in the past. We are aware that the finding of the feedback coupling is a rather complex problem because it is influenced by a lot of parameters which can change during the time evolution of the population. The parameters are fixed by the environment and all manipulations made in the past. Here, we consider a simple model where the growing rate r is replaced by a time changing growing rate $r(t)=r$

$\mp R(t)$, see Eq. (4) where $R(t)$ is a measure for all events in the past and depends on the population $p(t)$ in a self-consistent manner. Under the influence of this additional term the behavior of the system is changed totally. The different behavior is studied by analytical and numerical methods. The memory coupling is able to change the global behavior of the system in a nontrivial manner. Obviously, the paper is concerned with a toy model, containing a single degree of freedom and no spatial variations. However, some features as nonlinearity and a history-dependent accumulative term should be likewise included also in more complex models.

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